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A TOPOLOGICAL PROPERTY OF SUPERCOMPACT HAUSDORFF SPACES

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A topological property of supercompact Hausdorff spaces *)

by

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ABSTRACT

It is demonstrated that supercompactness in Hausdorff spaces implies a topological property which is not a property of all compact Hausdorff spaces. As an application it follows that an infinite compact F-space is not supercompact and consequently, for example, $\beta X \backslash X$ is not supercompact if X is a noncompact, locally compact and σ-compact topological space.

KEY WORDS & PHRASES: *Super compact, F-space, Čech-Stone compactification.*

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*) This paper is not for review; it is meant for publication elsewhere.

1. INTRODUCTION

In [4], DE GROOT defined a topological space X to be *supercompact* provided that it possesses an open subbase $\mathcal{B}$ such that each covering of X by elements of $\mathcal{B}$ contains a subcover of two elements of $\mathcal{B}$. Such a subbase is called *binary*. A supercompact space is compact; supercompactness is a topological invariant; it is productive; and the class of supercompact spaces contains the compact orderable spaces, compact tree-like spaces ([2],[5]) and compact polyhedra ([4]). Every topological space can be embedded, in a natural way, in many supercompact extensions, called superextensions. VERBEEK's thesis [6] is a good place to find the basic theorems about superextensions.

One of the main problems concerning supercompactness is the intriguing conjecture of DE GROOT [4] that all compact metric spaces are supercompact. This conjecture is still unsolved. DE GROOT in fact conjectured a stronger assertion: all compact Hausdorff spaces are supercompact. This is not the case. By a recent result of BELL [1], *if* X *is not pseudocompact, then* βX *is not supercompact*, a theorem which gives us many compact Hausdorff spaces which are not supercompact. The present paper is motivated by the following observation: as there are compact Hausdorff spaces which are not supercompact, supercompactness in Hausdorff spaces must imply a topological property which is not a property of all compact Hausdorff spaces. We will give an example of such a property. As an application it follows that if X is a non-compact, locally compact and σ-compact topological space, then $\beta X\backslash X$ is not supercompact. Moreover, we will show that if βX is a continuous image of a supercompact Hausdorff space, then X is pseudocompact, which generalizes Bell's theorem. We want to thank M. BELL for sending us a copy of the proof of the theorem cited above.

2. A TOPOLOGICAL PROPERTY OF SUPERCOMPACT HAUSDORFF SPACES

All topological spaces under discussion are assumed to be Tychonoff. Let X be a supercompact space. The supercompactness of X can also be described in terms of a closed subbase. If $\mathcal{B}$ is an open binary subbase for X, then $\mathcal{S} = \{X\backslash U \mid U \in \mathcal{B}\}$ is also called binary. The closed subbase $\mathcal{S}$ has the property that each linked subsystem $\mathcal{U}$ of $\mathcal{S}$ has a nonempty intersection, where linked means that every two members of $\mathcal{U}$ meet. We prefer to work with closed subbases.

LEMMA 1. *Let $\mathcal{S}$ be a binary closed subbase for* X. *Then for all* $x \in X$ *and for all* $S_0 \in \mathcal{S}$ *with* $x \notin S_0$, *there exists an* $S \in \mathcal{S}$ *such that* $x \in S$ *and* $S \cap S_0 = \emptyset$.

PROOF. Choose $S_0 \in \mathcal{S}$ and $x \in X$ such that $x \notin S_0$. Since X is T_1,

$$\{x\} = \bigcap \{S \in \mathcal{S} \mid x \in S\}$$

and consequently, since $\mathcal{S}$ is binary, there exists an $S \in \mathcal{S}$ such that $x \in S$ and $S \cap S_0 = \emptyset$. □

If B is a subset of X, then define

$$I(B) := \bigcap \{S \in \mathcal{S} \mid B \subset S\}.$$

Notice that $B \subset \bar{B} \subset I(B) = I(I(B))$. The following simple lemma will be used frequently.

LEMMA 2. *If* $A \subset B$, *then* $I(A) \subset I(B)$. *In particular, if* $A \subset I(B)$, *then* $I(A) \subset I(B)$. □

THEOREM 1. *Let* Y *be a continuous image of a supercompact space. If* Y *is infinite, then* Y *contains a copy of* $\mathbb{N}$ *that is not* C^**-embedded in* Y.

PROOF. Assume that there exists a continuous surjection $f: X \to Y$, such that X is a supercompact space with binary closed subbase $\mathcal{S}$. Moreover, assume that each copy of $\mathbb{N}$ in Y is C^*-embedded in Y. Choose a countable discrete subset $D \subset Y$ and choose for every $d \in D$ a $d' \in X$ such that $f(d') = d$. Let D' be the set of points obtained in this way. As D' is a countable discrete subset of X and since $\bar{D} = \beta D$ it follows that $\bar{D}' = \beta D'$ and that $f|\beta D'$ is a homeomorphism.

[A] Let $\mathcal{U} \in \beta D'$. Then $\{\mathcal{U}\} = \bigcap_{M\in\mathcal{U}} I(M)$. (We consider $\mathcal{U}$ to be an ultra-filter on D'.) Since for all $M \in \mathcal{U}$ we have $\mathcal{U} \in \bar{M} \subset I(M)$ it follows that in any case

$\mathcal{U} \in \bigcap_{M\in\mathcal{U}} I(M)$. Assume that there exists an $x \in \bigcap_{M\in\mathcal{U}} I(M)$ such that $x \neq \mathcal{U}$. Then, since $\mathcal{S}$ is a closed subbase, there exists an $S_0 \in \mathcal{S}$ such that $\mathcal{U} \in S_0$ and $x \notin S_0$. Now choose $S_1 \in \mathcal{S}$ such that $x \in S_1$ and $S_0 \cap S_1 = \emptyset$ (Lemma 1). Choose open U_i $(i=0,1)$ such that $S_i \subset U_i$ $(i=0,1)$ and $U_0 \cap U_1 = \emptyset$. Now, since $\mathcal{S}$ is a closed subbase and X is compact, there exist $S_{ij} \in \mathcal{S}$ and $S'_{ij} \in \mathcal{S}$ $(i,j=1,2,\ldots,n)$ such that

(i) $$X\setminus U_0 \subset \bigcup_{i=1}^{n} \bigcap_{j=1}^{n} S_{ij}; \qquad X\setminus U_1 \subset \bigcup_{i=1}^{n} \bigcap_{j=1}^{n} S'_{ij};$$

(ii) $$\bigcup_{i=1}^{n} \bigcap_{j=1}^{n} S_{ij} \cap S_0 = \emptyset = S_1 \cap \bigcup_{i=1}^{n} \bigcap_{j=1}^{n} S'_{ij}.$$

(Notice that a finite intersection of finite unions of subbase elements can also be represented as a finite union of finite intersections.) As

$$\bigcup_{i=1}^{n} \bigcap_{j=1}^{n} S_{ij} \cup \bigcup_{i=1}^{n} \bigcap_{j=1}^{n} S'_{ij} = X,$$

it follows that

$$\bigcup_{i=1}^{n} \left(\bigcap_{j=1}^{n} S_{ij}\cap D'\right) \cup \bigcup_{i=1}^{n} \left(\bigcap_{j=1}^{n} S'_{ij}\cap D'\right) = D'$$

and therefore, since $\mathcal{U}$ is an ultra-filter, at least one of the collection

$$\{A \mid A = \bigcap_{j=1}^{n} S_{ij} \cap D' \vee A = \bigcap_{j=1}^{n} S'_{ij} \cap D', \quad i \in \{1,2,\ldots,n\}\}$$

must belong to $\mathcal{U}$. If $\bigcap_{j=1}^{n} S_{ij}\cap D' \in \mathcal{U}$ for some $i \in \{1,2,\ldots,n\}$, then

$$\mathcal{U} \in \bigcap_{M\in\mathcal{U}} I(M) \cap S_0 \subset I\left(\bigcap_{j=1}^{n} S_{ij}\cap D'\right) \cap S_0 \subset \bigcap_{j=1}^{n} S_{ij} \cap S_0 = \emptyset,$$

which is a contradiction. If $\bigcap_{j=1}^{n} S'_{ij}\cap D' \in \mathcal{U}$ for some $i \in \{1,2,\ldots,n\}$, then

$$x \in \bigcap_{M\in\mathcal{U}} I(M) \cap S_1 \subset I\left(\bigcap_{j=1}^{n} S'_{ij}\cap D'\right) \cap S_1 \subset \bigcap_{j=1}^{n} S'_{ij} \cap S_1 = \emptyset,$$

which also is a contradiction.

[B] Choose $\mathcal{U} \in \beta D'$ and let $M = \{m_1, m_2, \ldots\} \in \mathcal{U}$. Then

$$\{\mathcal{U}\} = \bigcap_{k=1}^{\infty} I(\{\mathcal{U}, m_k\}) \cap I(M).$$

As $\mathcal{U} \in I(\{\mathcal{U}, m_k\})$ for all $k \in \mathbb{N}$ and since $\mathcal{U} \in \bar{M} \subset I(M)$ it follows that $\mathcal{U} \in \bigcap_{k=1}^{\infty} I(\{\mathcal{U}, m_k\}) \cap I(M)$. Assume that there exists an $x \in \bigcap_{k=1}^{\infty} I(\{\mathcal{U}, m_k\}) \cap I(M)$ such that $\mathcal{U} \neq x$. Then there exists an $M_0 \in \mathcal{U}$ such that $x \notin I(M_0)$ ([A]). Choose $m_{k_0} \in M \cap M_0$, then $\mathcal{U}, m_{k_0} \in \overline{M \cap M_0} \subset I(M \cap M_0) \subset I(M_0)$ and consequently $I(\{\mathcal{U}, m_{k_0}\}) \subset I(M_0)$ (Lemma 2), which is a contradiction since

$$x \in \bigcap_{k=1}^{\infty} I(\{\mathcal{U}, m_k\}) \cap I(M) \subset I(\{\mathcal{U}, m_{k_0}\}) \subset I(M_0).$$

[C] Choose $\mathcal{U} \in \beta D'$ and let $M \in \mathcal{U}$. Then there exists a *finite* subset $F(M)$ of M such that

$$\{f(\mathcal{U})\} = f[\bigcap_{k \in F(M)} I(\{\mathcal{U}, k\}) \cap I(F(M))].$$

(Actually, a notation such as $F_{\mathcal{U}}(M)$ would be better, as F depends on the ultra-filter $\mathcal{U}$. For reasons of notational simplicity we suppress the index $\mathcal{U}$.) Let $M = \{m_1, m_2, \ldots\}$ and define for each $k \in \mathbb{N}$

$$Z_k = \bigcap_{i=1}^{k} I(\{\mathcal{U}, m_i\}) \cap I(\{m_1, m_2, \ldots, m_k\}).$$

Notice that since $\mathcal{S}$ is binary, the set Z_k is nonvoid for all $k \in \mathbb{N}$. We will show that there exists a $k_0 \in \mathbb{N}$ such that $\{f(\mathcal{U})\} = f[Z_k]$ for all $k \geq k_0$. Suppose that this is not true; then for all $k \in \mathbb{N}$ there exists an $\ell \in \mathbb{N}$ with $\ell \geq k$ and $f[Z_\ell] \neq \{f(\mathcal{U})\}$. We will construct a sequence $\{x_n\}_{n=1}^{\infty}$ in Y such that

(i) $x_i \neq f(\mathcal{U})$ $(i \in \mathbb{N})$;

(ii) $x_i = x_j \Longleftrightarrow i = j$ $(i,j \in \mathbb{N})$;

(iii) for all $i \in \mathbb{N}$ there exists a $j \in \mathbb{N}$ with $j \geq i$ and $x_i \in f[Z_j]$.

Choose $k \in \mathbb{N}$ such that $f[Z_k] \neq \{f(\mathcal{U})\}$ and let $x_1 \in f[Z_k] \setminus \{f(\mathcal{U})\}$. Assume that all x_n have been constructed for $n \leq n_0$ $(n_0 \in \mathbb{N})$. Let $S \in \mathcal{S}$ such that $\mathcal{U} \notin S$. Then, since $\{\mathcal{U}\} = \bigcap_{k=1}^{\infty} I(\{\mathcal{U}, m_k\}) \cap I(M)$ ([B]) and $\mathcal{S}$ is binary, it follows that $I(M) \cap S = \emptyset$, or there exists a $k_0 \in \mathbb{N}$ such that $I(\{\mathcal{U}, m_{k_0}\}) \cap S = \emptyset$.

In the first case all $Z_k \cap S = \emptyset$ $(k \in \mathbb{N})$ and in the second case all $Z_k \cap S = \emptyset$ for $k \geq k_0$. Now, let O be an open neighbourhood of $f(\mathcal{U})$ such that $O \cap \{x_1, x_2, \ldots, x_{n_0}\} = \emptyset$. Choose $S_i \in \mathcal{S}$ $(i=1,2,\ldots,\ell)$ such that $\mathcal{U} \in X \backslash \bigcup_{i=1}^{\ell} S_i \subset f^{-1}[O]$. It is now obvious that there exists an index $k_1 \in \mathbb{N}$ such that $Z_k \subset f^{-1}[O]$ for all $k \geq k_1$ $(k \in \mathbb{N})$. Choose an index $k \in \mathbb{N}$ with $k \geq k_1$ such that $f[Z_k] \neq \{f(\mathcal{U})\}$, and let $x_{n_0+1} \in f[Z_k] \backslash \{f(\mathcal{U})\}$. This completes the construction of the x_n $(n \in \mathbb{N})$.

Now, since every open neighbourhood O of $\mathcal{U}$ contains all Z_k for k larger than or equal to some $k_0 \in \mathbb{N}$, it follows that for every open neighbourhood U of $f(\mathcal{U})$ the sequence $\{x_n\}_{n=1}^{\infty}$ is eventually in U. Therefore $\{x_n \mid n \in \mathbb{N}\} \cup \{f(\mathcal{U})\}$ is a convergent sequence, which contradicts the fact that each copy of $\mathbb{N}$ in Y is C^*-embedded in Y.

Therefore there exists a $k_0 \in \mathbb{N}$ such that $f[Z_k] = \{f(\mathcal{U})\}$ for all $k \geq k_0$. Now define

$$F(M) = \{m_1, m_2, \ldots, m_{k_0}\}.$$

Then

$$\{f(\mathcal{U})\} = f[\bigcap_{k \in F(M)} I(\{\mathcal{U},k\}) \cap I(F(M))].$$

[D] The contradiction. Let $A = \{E \subset D' \mid E \text{ is finite}\}$ and let $\mathcal{P}(A)$ be the powerset of A. Then $|\mathcal{P}(A)| = 2^{\aleph_0} = c$. Define a function

$$\xi \colon \beta D' \to \mathcal{P}(A)$$

by $\xi(\mathcal{U}) := \{F(M) \mid M \in \mathcal{U}\}$. Notice that $\xi(\mathcal{U}) \in \mathcal{P}(A)$ for all $\mathcal{U} \in \beta D'$. We will show that ξ is one-to-one. Choose $\mathcal{U}_0, \mathcal{U}_1 \in \beta D'$ such that $\mathcal{U}_0 \neq \mathcal{U}_1$ and choose open neighbourhoods U_i of $f(\mathcal{U}_i)$ $(i=0,1)$ such that $U_0 \cap U_1 = \emptyset$. Then $\mathcal{U}_i \in f^{-1}[U_i]$ $(i=0,1)$ and $f^{-1}[U_0] \cap f^{-1}[U_1] = \emptyset$. Suppose that for all $M \in \mathcal{U}_0$ we have $I(M) \cap (X \backslash f^{-1}[U_0]) \neq \emptyset$. Then the system $\{I(M) \mid M \in \mathcal{U}_0\} \cup \{X \backslash f^{-1}[U_0]\}$ has the finite intersection property, since if $M_i \in \mathcal{U}_0$ $(i=1,2,\ldots,n)$, then $\bigcap_{i=1}^{n} M_i \in \mathcal{U}_0$ and $I(\bigcap_{i=1}^{n} M_i) \subset \bigcap_{i=1}^{n} I(M_i)$, and therefore since X is compact it follows that $\mathcal{U} \in X \backslash f^{-1}[U_0]$ ([A]), which is a contradiction. Now choose $M_0 \in \mathcal{U}_0$ such that $\mathcal{U}_0 \in I(M_0) \subset f^{-1}[U_0]$. We will show that $F(M_0) \notin \{F(N) \mid N \in \mathcal{U}_1\}$. Assume to the contrary that there exists an $N_0 \in \mathcal{U}_1$ such that $F(M_0) = F(N_0)$. Then

$$\{f(\mathcal{U}_1)\} = f[\bigcap_{k\in F(N_0)} I\{(\mathcal{U}_1,k)\} \cap I(F(N_0))] \subset f[I(F(N_0))]$$

$$\subset f[I(F(M_0))] \subset f[I(M_0)] \subset U_0,$$

which is a contradiction. Therefore ξ is one-to-one. However, this is also a contradiction, since $|\beta D'| = 2^c$ ([3]). □

Notice that the above theorem implies that every infinite supercompact space X also contains a copy of $\mathbb{N}$ that is not C^*-embedded in X, since X is a continuous image of itself.

COROLLARY. *An infinite compact* F-*space is not supercompact.*

PROOF. Every countable subspace of an F-space is C^*-embedded (GILLMAN & JERISON [3], 14 N5). □

For every noncompact, locally compact and σ-compact topological space X, $\beta X\backslash X$ is an example of an infinite compact F-space ([3], 14.27) and consequently $\beta X\backslash X$ is not supercompact; it is not even the continuous image of a supercompact space. Another example is an infinite Gleason space, as was pointed out to me by M. BELL.

3. SUPERCOMPACTNESS IN βX

THEOREM 2. *If* βX *is a continuous image of a supercompact space, then* X *is pseudocompact.*

PROOF. Let Y be a supercompact space and f: Y $\to \beta X$ be a continuous surjection. Assume that Y has a binary closed subbase $\mathcal{S}$ and that X is not pseudocompact. Let Z_0 be a nonempty zero-set of βX, which has an empty intersection with X ([3]). Construct a countable discrete subset D of X such that $\bar{D}\backslash D \subset Z_0$. Then D is a closed subspace of $X^* = \beta X\backslash Z_0$ and as X^* is σ-compact, and hence normal, D is C^*-embedded in $\beta X^* = \beta X$. Therefore $\bar{D} = \beta D$. For every $d \in D$, choose $d' \in Y$ such that $f(d') = d$ and let D' be the set of points obtained in this way. Then D' is also a countable discrete subspace of Y and it is obvious that $\bar{D}' = \beta D'$.

[A] Let $\mathcal{U} \in \beta D'$. Then $\{\mathcal{U}\} = \bigcap_{M\in\mathcal{U}} I(M)$.

[B] Let $\mathcal{U} \in \beta D'$ and let $M = \{m_1, m_2, \ldots\} \in \mathcal{U}$. Then

$$\{\mathcal{U}\} = \bigcap_{k=1}^{\infty} I(\{\mathcal{U}, m_k\}) \cap I(M).$$

The proofs of [A] and [B] are the same as the proofs of [A] and [B] of Theorem 1.

[C] Choose $\mathcal{U} \in \beta D' \backslash D'$ and let $M \in \mathcal{U}$. Then there exists a finite subset $F(M)$ of M such that

$$\{f(\mathcal{U})\} = f[\bigcap_{k\in F(M)} I(\{\mathcal{U}, k\}) \cap I(F(M))].$$

Indeed, let $M = \{m_1, m_2, \ldots\}$ and define as in Theorem 1

$$Z_k = \bigcap_{i=1}^{k} I(\{\mathcal{U}, m_i\}) \cap I(\{m_1, m_2, \ldots, m_k\}).$$

If for all $k \in \mathbb{N}$ there exists an $\ell \in \mathbb{N}$ with $\ell \geq k$ and $f[Z_k] \neq \{f(\mathcal{U})\}$, then in the same way as in Theorem 1 we can construct a sequence $\{x_n\}_{n=1}^{\infty}$ in βX such that $\{f(\mathcal{U})\} \cup \{x_n \mid n \in \mathbb{N}\}$ is a convergent sequence. It is clear that $\{x_n \mid n \in \mathbb{N}\} \cap X^*$ is closed in X^*, consequently is C^*-embedded in X^*, and therefore is finite. Now, $\{x_n \mid n \in \mathbb{N}\} \cap Z_0$ is infinite and as Z_0 is an F-space ([3], 14.27; $Z_0 = \beta X^* \backslash X^*$) this is a contradiction, since $f(\mathcal{U}) \in Z_0$. Therefore there exists a $k_0 \in \mathbb{N}$ such that $f[Z_k] = \{f(\mathcal{U})\}$ for all $k \geq k_0$. Now define $F(M) = \{m_1, m_2, \ldots, m_{k_0}\}$.

[D] In practically the same way as in Theorem 1 we can derive a contradiction. □

It now follows that for example $\beta\mathbb{N}$ is not a retract of one of its Hausdorff superextensions. In particular $\beta\mathbb{N}$ is not a retract of $\lambda\mathbb{N}$. This theorem also gives a new proof of the well-known fact: βX dyadic implies that X is pseudocompact.

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